

# A “Hot-Spot” Proof of Normality for the Alpha Constants

David H. Bailey

24 April 2003

In [2] Richard Crandall and I establish normality base  $p$  for the class of constants

$$\alpha_{p,q} = \sum_{k=1}^{\infty} \frac{1}{q^k p^{q^k}}$$

where  $p$  and  $q$  are co-prime. The proof given in [2] is somewhat difficult and relies on several not-well-known results, including one by Korobov on the properties of certain pseudo-random sequences. In this note I show that normality can be established much more easily, as a consequence of what is sometimes call the “hot spot” theorem. The “hot spot” theorem is as follows. In the following,  $\{\cdot\}$  denotes fractional part.

**“Hot Spot” Theorem.** The real constant  $\alpha$  is normal base  $b$  if and only if there exists a constant  $C$  such that for every subinterval  $[c, d) \subset [0, 1)$ ,

$$\limsup_{n \geq 1} \frac{\#\{0 \leq j < n \mid \{b^j \alpha\} \in [c, d)\}}{n} \leq C(d - c).$$

In other words, normal numbers have no “hot spots”, and conversely a non-normal number must have hot spots — there must be digit strings that appear, say, one billion times more often than the frequency they would appear if the number were normal. A proof of the hot spot theorem, using the Birkhoff ergodic theorem [3, pg. 13, 20-29], is given in [1], where it is used to establish that a rational times a normal number is normal. The hot spot theorem is proved by a different (but more difficult) argument in [4, pg. 77].

Here is how the hot spot theorem can be used to establish normality for the  $\alpha$  constants studied in [2]. In this note I will use  $\alpha = \alpha_{2,3}$ , namely

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k}},$$

but the proof is very similar for other  $\alpha_{p,q}$  constants from [2].

**Theorem.**  $\alpha$  is normal base 2.

**Proof:** First note that the associated sequence (in the BBP sense) for  $\alpha$  is  $x_0 = 0$ , with  $x_n = \{2x_{n-1} + r_n\}$ , where  $r_n = 1/n$  if  $n = 3^k$ , and zero otherwise. As above, the notation  $\{\cdot\}$  denotes fractional part. Observe that the  $x$  sequence has the pattern

$$\begin{aligned} &0, 0, 0, 1/3, 2/3, 1/3, 2/3, 1/3, 2/3, \\ &4/9, 8/9, 7/9, 5/9, 1/9, 2/9, 4/9, 8/9, 7/9, 5/9, 1/9, 2/9, \end{aligned}$$

and so forth. Note that for  $n < 3^{k+1}$ , each  $x_n$  is a multiple of  $1/3^k$ , and each fraction  $j/3^k$ ,  $0 \leq j < 3^k$  appears exactly three times in the sequence. Also note that

$$|x_n - \alpha_n| = \left| \sum_{k=n+1}^{\infty} 2^{n-k} r_k \right| < \frac{1}{2n}$$

where  $\alpha_n = \{2^n \alpha\}$  is the tail of the binary expansion of  $\alpha$  after the first  $n$  bits.

Given  $n$ , let  $m$  be the largest power of 3 less than  $n$ , and assume that  $n$  is large enough so that  $n > m > 1/(d-c)$ . Now note that the interval  $[c - 1/(2n), d + 1/(2n))$  contains exactly  $m(d-c)$  (or possibly one more than this number) multiples of  $1/m$ , and thus can contain at most three times this many occurrences of  $x_j$  in the first  $n$  elements, assuming  $n > m$ . Thus for  $n > m$  we have

$$\begin{aligned} \frac{\#_{0 \leq j < n}(\alpha_j \in [c, d])}{n(d-c)} &\leq \frac{\#_{0 \leq j < n}(x_j \in [c - 1/(2n), d + 1/(2n)])}{n(d-c)} \\ &\leq \frac{3[m(d-c) + 1]}{n(d-c)} < \frac{3[m(d-c) + 1]}{m(d-c)} \\ &= 3 + \frac{3}{m(d-c)} < 6 \end{aligned}$$

In other words, for all  $n > 3(d-c)$ , say, no subinterval of  $[0, 1)$  has more than six times as many elements of  $x_j$  as it “should” in the first  $n$  elements, and no binary string appears more than six times as often as it “should” in first  $n$  bits of the binary expansion of  $\alpha$ . By the hot spot theorem, this establishes that  $\alpha$  is normal base 2.

## References

- [1] David H. Bailey and Daniel J. Rudolph, “An Ergodic Proof that Rational Times Normal is Normal,” manuscript, available at the URL <http://www.nersc.gov/~dhbailey/dhbpapers/ratxnormal.pdf>
- [2] David H. Bailey and Richard E. Crandall, “Random Generators and Normal Numbers,” to appear in *Experimental Mathematics*, available at the URL <http://www.nersc.gov/~dhbailey/dhbpapers/bcnormal.pdf>
- [3] Patrick Billingsley, *Ergodic Theory and Information*, John Wiley, New York, 1965.
- [4] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley-Interscience, New York, 1974.